

# On the Optimal Solutions of the Infinite-Horizon Linear Sensor Scheduling Problem

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**Abstract**—This paper studies the infinite-horizon sensor scheduling problem for linear Gaussian processes with linear measurement functions. Several important properties of the optimal infinite-horizon schedules are derived. In particular, it is proved that under some mild conditions, both the optimal infinite-horizon average-per-stage cost and the corresponding optimal sensor schedules are independent of the covariance matrix of the initial state. It is also proved that the optimal estimation cost can be approximated arbitrarily closely by a periodic schedule with a finite period, and moreover, the trajectory of the error covariance matrix under this periodic schedule converges exponentially to a unique limit cycle. These theoretical results provide valuable insights about the problem and can be used as general guidelines in the design and analysis of various infinite-horizon sensor scheduling algorithms.

## I. INTRODUCTION

The sensor scheduling problem tries to find a schedule over a certain time horizon to activate/deactivate a subset of available sensors to improve the estimation performance and reduce the estimation cost (e.g. energy consumption and communication overheads). It has numerous applications in various engineering fields [1], [2], [3].

Previous research has mainly focused on the finite-horizon sensor scheduling problem for linear Gaussian processes with linear measurement functions. In this case, for a given sensor schedule, the optimal state estimate can be obtained using the Kalman filter and the corresponding error covariance matrix can be computed recursively using the difference Riccati recursion. Thus, a straightforward way to solve this scheduling problem is to enumerate all the possible finite-horizon schedules [1]. The complexity of such an approach grows exponentially fast as the horizon length increases. Various methods have been proposed in the literature to tackle this challenge. These methods can be roughly divided into the following three categories: (i) methods that focus on certain simple special classes of schedules, such as myopic schedules that only consider immediate performance at each time step instead of the overall performance over the whole horizon [4], [5]; (ii) methods that “embed” discrete schedules into a larger class of schedules with continuously-variable sensor indices [6], [7]; (iii) and methods that prune the search

tree based on certain properties of the Riccati recursions [8], [9].

The methods in the first category are often easy to implement, but provide no guarantees for the overall estimation performance. The “embedding” approach in the second category is a common trick to tackle complex discrete optimization or optimal control problems [10], [11]. The resulting relaxed schedule can often be interpreted as the time-average “frequencies” or “probabilities” for using different sensors. It has been recently proved [7] that, in continuous time, the performance of the optimal relaxed schedule can be approximated with arbitrary accuracy by a discrete schedule through fast switchings. This is analogous to the result derived in [11] for solving the optimal control problem of switched systems using embedding. However, in discrete time, the result no longer holds as the switching rate is fixed; in this case, the relaxed schedule can only be implemented probabilistically [6], resulting in a random scheduling of the sensors with random error performances. The pruning methods in the third category make essential use of the monotonicity and concavity properties of the Riccati mapping (See Lemma 1) to obtain conditions under which the exploration of certain branches can be avoided without losing the optimal schedule. In our earlier paper [8], an efficient algorithm was proposed to prune out not only the non-optimal branches but also less important ones to further reduce the complexity. Some error bounds associated with this pruning algorithm have also been derived in [12].

In recent years, the sensor scheduling problem for nonlinear stochastic systems with nonlinear measurement functions have also been extensively studied [3], [13], [14]. The problem is often formulated as a Markov decision problem and solved using dynamic programming, where the value functions are computed either through gridding the state space or through sampling the state space using Monte Carlo simulations. The approach applies to virtually all types of dynamical processes, but its complexity is prohibitive for high state dimensions.

Different from most previous research, this paper studies the infinite-horizon sensor scheduling problem for discrete-time linear Gaussian processes observed by linear sensors. The problem is much more challenging than its finite-horizon counterpart and has not been adequately investigated in the literature. Instead of proposing a specific scheduling algorithm, we focus on deriving several fundamental properties of the problem that can be used as general guidelines in the design and analysis of various infinite-horizon sensor scheduling algorithms. In particular, it is proved that un-

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der some mild conditions, both the optimal infinite-horizon average-per-stage cost and the corresponding optimal sensor schedule are independent of the covariance matrix of the initial state. It is also proved that the optimal estimation cost can be approximated arbitrarily closely by a periodic schedule with a finite period, and moreover, the trajectory of the error covariance matrix under this periodic schedule converges exponentially fast to a unique limit cycle, regardless of the initial covariance matrix. These theoretical properties provide us valuable insight into the infinite-horizon sensor scheduling problem and will be useful for developing algorithms. In addition, the existence of a periodic suboptimal schedule justifies the experimental results of many finite-horizon scheduling algorithms [8], [15] that yield periodic schedules for relatively large horizons.

The rest of the paper is organized as follows. The infinite-horizon sensor scheduling problem is formulated in Section II. Some important properties of the difference Riccati recursion are derived in Section III. These properties are then used in Section IV to prove various properties of the optimal solutions of the infinite-horizon sensor scheduling problem. Finally, some concluding remarks are given in Section V.

**Notation:** Let  $\mathcal{A}$  be the semi-definite cone, namely, the set of all the positive semidefinite matrices. Denote by  $\lambda_{\min}(\cdot)$  and  $\lambda_{\max}(\cdot)$  the smallest and the largest eigenvalues, respectively, of a given matrix in  $\mathcal{A}$ . Let  $\mathbb{R}_+$  and  $\mathbb{Z}_+$  be the set of nonnegative real numbers and integers, respectively. Let  $\|\cdot\|$  be the standard Euclidean norm of vectors as well as the corresponding induced norm of matrices. Denote by  $|\cdot|$  the cardinality of a given set. For any  $\phi_c \in \mathcal{A}$  and  $r > 0$ , define  $\mathcal{B}(\phi_c; r) := \{\phi \in \mathcal{A} : \|\phi - \phi_c\| \leq r\}$ . Denote by  $I_n$  the identity matrix of dimension  $n$ .

## II. PROBLEM FORMULATION

Consider the following linear time-invariant stochastic system:

$$x(t+1) = Ax(t) + w(t), \quad t \in \mathbb{Z}_+, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state of the system and  $w(t)$  is the process noise. The initial state,  $x(0)$ , is assumed to be Gaussian with zero mean and covariance matrix  $\phi_0$ , i.e.,  $x(0) \sim \mathcal{N}(0, \phi_0)$ . There are  $M$  different sensors attached to the process. At each time step, we assume that only one of the  $M$  sensors is available to take measurements. The measurement of the  $i^{\text{th}}$  sensor is given by:

$$y_i(t) = C_i x(t) + v_i(t), \quad t \in \mathbb{Z}_+, \quad (2)$$

where  $y_i(t) \in \mathbb{R}^p$  and  $v_i(t) \in \mathbb{R}^p$  are the measurement output and measurement noise of the  $i^{\text{th}}$  sensor at time  $t$ , respectively. We assume that the process noise and all the measurement noises are mutually independent Gaussian white noises given by:

$$w(t) \sim \mathcal{N}(0, \Phi^w), \quad v_i(t) \sim \mathcal{N}(0, \Phi_i^v).$$

Define  $\lambda_w^- = \lambda_{\min}(\Phi^w)$  and  $\lambda_v^- = \min_{i \in \mathbb{M}} \{\lambda_{\min}(\Phi_i^v)\}$ . Assume that  $\lambda_w^- > 0$  and  $\lambda_v^- > 0$ . Let  $\mathbb{M} := \{1, \dots, M\}$  be the set of sensor indices. For each  $N \in \mathbb{Z}_+$ , denote by  $\mathbb{M}^N$

the set of all the sequences of sensor indices of length  $N$ . An element  $\sigma \in \mathbb{M}^N$  is called an  $N$ -horizon *sensor schedule*. The set of all infinite-horizon sensor schedules is denoted by  $\mathbb{M}^\infty$ . An infinite-horizon schedule  $\sigma \in \mathbb{M}^\infty$  is called periodic with a period  $l \in \mathbb{Z}_+$  if  $\sigma(t) = \sigma(t+l)$  for all  $t \in \mathbb{Z}_+$ . Under a given sensor schedule  $\sigma \in \mathbb{M}^\infty$ , the measurement sequence is determined by:

$$y(t) = y_{\sigma(t)}(t) = C_{\sigma(t)} x(t) + v_{\sigma(t)}(t), \quad \forall t \in \mathbb{Z}_+.$$

For each  $t_1 \leq t_2 < \infty$ , denote by  $\hat{x}^\sigma(t_2|t_1)$  the minimum mean-square error (MMSE) estimate of  $x(t_2)$  given the measurements  $\{y(0), \dots, y(t_1)\}$ , the initial covariance  $\phi_0$  and the sensor schedule  $\sigma \in \mathbb{M}^\infty$ . Define the predictor error  $e^\sigma(t|t-1)$  by

$$e^\sigma(t|t-1) = x(t) - A\hat{x}^\sigma(t-1|t-1),$$

and let  $\Sigma_t^\sigma(\phi_0)$  be its covariance matrix. When no ambiguity arises, we may drop its dependence on the initial covariance matrix and simply write  $\Sigma_t^\sigma$ . For any  $Q \in \mathcal{A}$ , denote by  $K_i(Q)$  the Kalman gain associated with sensor  $i \in \mathbb{M}$  and matrix  $Q$ , which is given by:

$$K_i(Q) = QC_i^T (C_i QC_i^T + \Phi_i^v)^{-1}. \quad (3)$$

By a standard result of the linear estimation theory, the error covariance of the predictor can be updated recursively using the Riccati recursion:

$$\begin{aligned} \Sigma_{t+1}^\sigma &= \Phi^w + A\Sigma_t^\sigma A^T \\ &\quad - A\Sigma_t^\sigma C_{\sigma(t)}^T \left( C_{\sigma(t)} \Sigma_t^\sigma C_{\sigma(t)}^T + \Phi_{\sigma(t)}^v \right)^{-1} C_{\sigma(t)} \Sigma_t^\sigma A^T. \end{aligned} \quad (4)$$

For each  $i \in \mathbb{M}$  and  $Q \in \mathcal{A}$ , define

$$\bar{A}_i(Q) \triangleq A - AK_i(Q)C_i. \quad (5)$$

Then, the Riccati recursion (4) can also be written as ([16])

$$\begin{aligned} \Sigma_{t+1}^\sigma &= \Phi^w + [\bar{A}_{\sigma(t)}(\Sigma_t^\sigma)] \cdot \Sigma_t^\sigma \cdot [\bar{A}_{\sigma(t)}(\Sigma_t^\sigma)]^T \\ &\quad + A \cdot [K_{\sigma(t)}(\Sigma_t^\sigma)] \Phi_{\sigma(t)}^v \cdot [K_{\sigma(t)}(\Sigma_t^\sigma)]^T \cdot A^T. \end{aligned} \quad (6)$$

For any finite integer  $N$ , the performance of an  $N$ -horizon sensor schedule  $\sigma \in \mathbb{M}^N$  can be evaluated according to the total estimation error defined by:

$$J_N(\sigma; \phi_0) \triangleq \sum_{t=1}^N \text{tr}(\Sigma_t^\sigma(\phi_0)), \quad (7)$$

or according to the average-per-stage estimation error defined by:

$$\bar{J}_N(\sigma; \phi_0) \triangleq \frac{1}{N} J_N(\sigma; \phi_0). \quad (8)$$

Clearly, whenever  $N$  is finite, the two cost functions  $J_N$  and  $\bar{J}_N$  are equivalent in the sense that they produce the same set of optimal solutions. However, the total cost  $J_N(\sigma; \phi_0) \rightarrow \infty$  as  $N \rightarrow \infty$  for all  $\sigma \in \mathbb{M}^\infty$  and  $\phi_0 \in \mathcal{A}$  because the system is constantly perturbed by a nontrivial Gaussian noise  $w(t)$ . Thus, the performance of an infinite-horizon sensor schedule

is usually measured by the limsup of the  $N$ -horizon average-per-stage cost:

$$\bar{J}_\infty(\sigma; \phi_0) \triangleq \limsup_{N \rightarrow \infty} \bar{J}_N(\sigma; \phi_0).$$

This cost function has been extensively used for studying various infinite-horizon optimal control and estimation problems [7], [17]. However, this cost function depends only on the limiting behavior of the schedule, which may lead to rather abnormal optimal solutions. For example, one can manipulate a finite portion of an optimal schedule to create an arbitrary transient behavior for the error trajectory without affecting the optimality of the schedule. In some extreme cases, the optimal schedule may even have an unbounded error covariance while still resulting in the minimum infinite-horizon average error. To exclude these abnormalities for the infinite horizon, we introduce the following feasible set of sensor schedules with bounded peak covariance:

$$\mathbb{M}_\phi^\infty = \{\sigma \in \mathbb{M}^\infty : \exists \beta < \infty, \text{ s.t.} \\ \Sigma_t^\sigma(\phi) \preceq \beta I_n, \forall t \in \mathbb{Z}_+, \phi \in \mathcal{A}\}.$$

For an arbitrary matrix  $\phi \in \mathcal{A}$ , an infinite-horizon sensor schedule  $\sigma$  is called *feasible* for  $\phi$  if  $\sigma \in \mathbb{M}_\phi^\infty$ . The following assumption is adopted throughout this paper.

*Assumption 1:*  $\mathbb{M}_\phi^\infty \neq \emptyset, \forall \phi \in \mathcal{A}$ .

*Remark 1:* The assumption requires that for any initial covariance, there always exists an infinite-horizon schedule that can keep the estimation error bounded for all time. This is a reasonable assumption for typical estimation applications. It can be guaranteed if, for example, one of the subsystems is detectable.

*Problem 1:* For a given  $\phi_0 \in \mathcal{A}$ , solve the following problem

$$\bar{V}^*(\phi_0) \triangleq \inf_{\sigma \in \mathbb{M}_{\phi_0}^\infty} \limsup_{N \rightarrow \infty} \bar{J}_N(\sigma; \phi_0) \quad (9)$$

Assumption 1 implies that  $\bar{V}^*(\phi_0)$  is finite for all  $\phi_0 \in \mathcal{A}$ . The function  $\bar{V}^* : \mathcal{A} \rightarrow \mathbb{R}_+$  defined implicitly by equation (9) is called the optimal infinite-horizon (average-per-stage) cost function. For a general  $\phi \in \mathcal{A}$ , a schedule that achieves the cost  $\bar{V}^*(\phi)$  will be referred to as an *optimal schedule* for  $\phi$ .

### III. SEQUENTIAL RICCATI MAPPING AND ITS STABILITY

The Riccati recursion in (4) can be viewed as a mapping that maps a given matrix  $\Sigma_t^\sigma \in \mathcal{A}$  to another matrix  $\Sigma_{t+1}^\sigma \in \mathcal{A}$  depending on the sensor index chosen at time  $t$ . In general, for each sensor  $i \in \mathbb{M}$ , we can define the *Riccati mapping* as

$$\rho_i(Q) = \Phi^w + AQA^T \\ - AQC_i^T (C_iQC_i^T + \Phi_i^v)^{-1} C_iQA^T, \forall Q \in \mathcal{A}. \quad (10)$$

With this notation, for a generic initial covariance matrix  $\phi \in \mathcal{A}$ , the covariance matrix  $\Sigma_t^\sigma(\phi)$ , defined in (4), is the trajectory of the following matrix-valued time-varying nonlinear system:

$$\Sigma_{t+1}^\sigma = \rho_{\sigma(t)}(\Sigma_t^\sigma), \text{ for } t \in \mathbb{Z}_+, \text{ with } \Sigma_0^\sigma = \phi. \quad (11)$$

One can also view  $\Sigma_t^\sigma(\cdot)$  as the composition of a sequence of Riccati mappings, i.e.,

$$\Sigma_t^\sigma = \rho_{\sigma(t-1)} \circ \rho_{\sigma(t-2)} \cdots \circ \rho_{\sigma(0)}, \quad t \in \mathbb{Z}_+. \quad (12)$$

To solve Problem 1, it is critical to understand the dynamical behavior of the matrix-valued nonlinear system (11) under different infinite-horizon schedules. Two well-known properties of the Riccati mapping are useful for this purpose.

*Lemma 1:* For any  $i \in \mathbb{M}$ ,  $Q_1, Q_2 \in \mathcal{A}$  and  $c \in [0, 1]$ , we have

- (i)  $Q_1 \preceq Q_2 \Rightarrow \rho_i(Q_1) \preceq \rho_i(Q_2)$ ;
- (ii)  $\rho_i(cQ_1 + (1-c)Q_2) \succeq c\rho_i(Q_1) + (1-c)\rho_i(Q_2)$ .

*Remark 2:* The lemma indicates that the Riccati mapping is *monotone* and *concave*. The monotonicity property is a well-known result and its proof can be found in [18]. The concavity property is an immediate consequence of Lemma 1-(e) in [19].

Based on these two properties, one can prove the following results.

*Theorem 1:* For any  $\phi \in \mathcal{A}$ ,  $\epsilon \in \mathbb{R}^+$ ,  $\sigma \in \mathbb{M}^\infty$ , and  $t \in \mathbb{Z}_+$ , we have  $\Sigma_t^\sigma(\phi + \epsilon I_n) \preceq \Sigma_t^\sigma(\phi) + g_t^\sigma(\phi) \cdot \epsilon$ . Furthermore, if  $\Sigma_t^\sigma(\phi) \preceq \beta I_n$  for all  $t \in \mathbb{Z}_+$  and some  $\beta < \infty$ , then  $\text{tr}(g_t^\sigma(\phi)) \leq n\beta/\lambda_w^- \eta^t, \forall t \in \mathbb{Z}_+$ , where

$$\eta = \frac{1}{1 + \alpha \lambda_w^-} < 1 \quad \text{and} \quad \alpha = \frac{\lambda_w^-}{\|A\|^2 \beta^2 + \lambda_w^- \beta}. \quad (13)$$

*Proof:* See [20]. ■

The above theorem reveals an important property of system (11), namely, boundedness of the trajectory implies an exponential disturbance attenuation. This property plays a crucial role in deriving the various properties of the optimal infinite-horizon schedules in Section IV.

### IV. PROPERTIES OF OPTIMAL SCHEDULE

In this section, we will use the properties of the sequential Riccati mapping derived in the last section to gain some insights on the optimal solutions of Problem 1.

#### A. Independence of Initial Covariance

We first show that the feasible set is independent of the initial covariance.

*Lemma 2:* If  $\sigma \in \mathbb{M}_{\phi_1}^\infty$  for some  $\phi_1 \in \mathcal{A}$ , then  $\sigma \in \mathbb{M}_\phi^\infty$  for all  $\phi \in \mathcal{A}$ .

*Proof:* Fix arbitrary  $\phi_1 \in \mathcal{A}$ ,  $\phi \in \mathcal{A}$  and  $\sigma \in \mathbb{M}_{\phi_1}^\infty$ . Since  $\phi \preceq \phi_1 + \|\phi - \phi_1\|I_n$ , by Theorem 1, we have

$$\Sigma_t^\sigma(\phi) \preceq \Sigma_t^\sigma(\phi_1) + g_t^\sigma(\phi_1) \cdot \|\phi - \phi_1\|.$$

The first term on the right hand side is bounded because  $\sigma \in \mathbb{M}_{\phi_1}^\infty$ , while the second term is bounded due to Theorem 1. Thus,  $\sigma \in \mathbb{M}_\phi^\infty$ . ■

Therefore, if an infinite-horizon schedule is feasible for some initial covariance matrix, it will be feasible for all initial covariances. This allows us to drop the dependence of the feasible set on the initial covariance and simply define

$$\mathbb{M}_f^\infty = \{\sigma \in \mathbb{M}^\infty : \exists \beta < \infty, \phi \in \mathcal{A}, \\ \text{s.t. } \Sigma_t^\sigma(\phi) \preceq \beta I_n, \forall t \in \mathbb{Z}_+\}. \quad (14)$$

We next show that under a fixed schedule  $\sigma \in \mathbb{M}_f^\infty$ , all the trajectories starting from different initial covariances will eventually converge to the same trajectory.

*Theorem 2:* For any feasible schedule  $\sigma \in \mathbb{M}_f^\infty$ , we have

$$\|\Sigma_t^\sigma(\phi_1) - \Sigma_t^\sigma(\phi_2)\| \rightarrow 0 \text{ exponentially as } t \rightarrow \infty,$$

for all  $\phi_1, \phi_2 \in \mathcal{A}$ .

*Proof:* Fix arbitrary  $\phi_1 \in \mathcal{A}$  and  $\phi_2 \in \mathcal{A}$ . Define  $\epsilon = \|\phi_1 - \phi_2\|$ . Without loss of generality, let  $\beta < \infty$  be the bound such that  $\Sigma_t^\sigma(\phi_i) \preceq \beta I_n$  for all  $t \in \mathbb{Z}_+$  and  $i = 1, 2$ . By Theorem 1, we have

$$\begin{aligned} \Sigma_t^\sigma(\phi_2) &\preceq \Sigma_t^\sigma(\phi_1 + \|\phi_2 - \phi_1\|I_n) \\ &\preceq \Sigma_t^\sigma(\phi_1) + g_t^\sigma(\phi_1) \cdot \epsilon \\ &\preceq \Sigma_t^\sigma(\phi_1) + \left(\frac{n\beta\epsilon}{\lambda_w} \eta^t\right) \cdot I_n. \end{aligned} \quad (15)$$

Similarly, we can obtain

$$\Sigma_t^\sigma(\phi_1) \preceq \Sigma_t^\sigma(\phi_2) + \left(\frac{n\beta\epsilon}{\lambda_w} \eta^t\right) \cdot I_n,$$

for all  $t \in \mathbb{Z}_+$ . The result follows directly from the above inequalities as  $t \rightarrow \infty$ .  $\blacksquare$

An immediate consequence of the above theorem is that the infinite-horizon average-per-stage cost of any feasible schedule is independent of the initial covariance matrix.

*Corollary 1:* For any  $\sigma \in \mathbb{M}_f^\infty$ ,  $\bar{J}_\infty(\sigma; \phi_1) = \bar{J}_\infty(\sigma; \phi_2)$  for all  $\phi_1, \phi_2 \in \mathcal{A}$ .

*Proof:* By Theorem 2,  $\Sigma_t^\sigma(\phi_1) \rightarrow \Sigma_t^\sigma(\phi_2)$  as  $t \rightarrow \infty$ . Thus, the two sequences  $\{\frac{1}{N} \sum_{t=1}^N \Sigma_t^\sigma(\phi_i)\}_{N \in \mathbb{Z}_+}$ ,  $i = 1, 2$ , must have the same limsup.  $\blacksquare$

By the above corollary, it is easy to see that if a feasible schedule  $\sigma$  is optimal for some initial covariance  $\phi_1$ , then it must also be optimal for any other initial covariance  $\phi_2$ . In addition, the optimal infinite-horizon average-per-stage costs corresponding to these two initial covariances must also be the same.

*Corollary 2:* For any  $\phi_1, \phi_2 \in \mathcal{A}$ , if  $\sigma^*$  is optimal for  $\phi_1$ , then it must also be optimal for  $\phi_2$ ; and in addition,  $\bar{V}^*(\phi_1) = \bar{V}^*(\phi_2)$ .

Therefore, to solve Problem 1, we can start from any initial covariance matrix at our convenience. The obtained optimal solution would also be optimal for all the other initial covariances.

## B. Stable Accumulation Sets Under Feasible Schedules

For any  $\sigma \in \mathbb{M}_f^\infty$ , let  $\mathcal{L}^\sigma$  be the *accumulation set* of the closed-loop trajectory of the nonlinear system (11) under schedule  $\sigma$  with a zero initial covariance. In other words, the set  $\mathcal{L}^\sigma$  contains all the points whose arbitrary neighborhoods will be visited infinitely often by the trajectory  $\{\Sigma_t^\sigma(0)\}_{t \in \mathbb{Z}_+}$ . This set characterizes the dynamical behavior of system (11) under the schedule  $\sigma$ .

According to Theorem 2, a trajectory  $\{\Sigma_t^\sigma(\phi)\}_{t \in \mathbb{Z}_+}$  under schedule  $\sigma$  starting from any initial covariance  $\phi \in \mathcal{A}$  will converge to the same accumulation set  $\mathcal{L}^\sigma$ . This implies the global attractiveness of the accumulation set.

*Theorem 3:* The accumulation set is globally asymptotically stable, i.e.,  $\Sigma_t^\sigma(\phi) \rightarrow \mathcal{L}^\sigma$  as  $t \rightarrow \infty$ , for all  $\phi \in \mathcal{A}$ .

*Proof:* Follows directly from the definition of the accumulation set and Theorem 2.  $\blacksquare$

## C. Periodic Suboptimal Schedule

The goal of this subsection is to show that the optimal infinite-horizon cost can be approximated with an arbitrary accuracy by a periodic schedule. Throughout this subsection, unless otherwise stated, we will denote by  $\sigma$  an arbitrary feasible schedule in  $\mathbb{M}_f^\infty$ , by  $\hat{\phi}$  an arbitrary accumulation point in  $\mathcal{L}^\sigma$ , by  $c$  an arbitrary constant in  $(0, 1)$  and by  $r$  an arbitrary positive finite constant. In addition, for any  $j \in \mathbb{Z}_+$ , let  $\sigma_{j+}$  be another infinite-horizon schedule obtained by removing the first  $j$  steps from  $\sigma$ , i.e.,  $\sigma_{j+} = \{\sigma(j), \sigma(j+1), \dots\}$ .

*Lemma 3 (Uniform Bound):* For any bounded set  $E \subset \mathcal{A}$ , there exists finite constants  $\beta_E, \alpha_E$  and  $\eta_E \in (0, 1)$  such that  $\Sigma_t^{\sigma_{j+}}(\phi) \preceq \beta_E I_n$  and  $\text{tr}(g_t^{\sigma_{j+}}(\phi)) \leq \alpha_E \eta_E^t$ , for all  $j, t \in \mathbb{Z}_+$  and  $\phi \in E$ .

*Proof:* Fix an arbitrary  $\phi_1 \in E$ . Define the covariance trajectory under  $\sigma$  with initial covariance  $\phi_1$  as  $\psi_t = \Sigma_t^\sigma(\phi_1)$ ,  $t \in \mathbb{Z}_+$ . Since  $\sigma$  is feasible, there must exist a finite constant  $\beta_1$  such that  $\psi_t \preceq \beta_1 I_n$  for all  $t \in \mathbb{Z}_+$ . By Theorem 1, there exist constants  $\alpha_1 < \infty$  and  $\eta_1 \in (0, 1)$  such that  $\text{tr}(g_t^\sigma(\phi_1)) \leq \alpha_1 \eta_1^t$ , for all  $t \in \mathbb{Z}_+$ . It can be easily verified that for any  $t, j \in \mathbb{Z}_+$ , we have  $\Sigma_t^{\sigma_{j+}}(\psi_j) = \psi_{t+j}$ . Thus,  $\text{tr}(g_t^{\sigma_{j+}}(\psi_j)) \leq \alpha_1 \eta_1^t$  as well for all  $t, j \in \mathbb{Z}_+$ . Therefore, by Theorem 1,

$$\begin{aligned} \Sigma_t^{\sigma_{j+}}(\phi) &\preceq \Sigma_t^{\sigma_{j+}}(\psi_j + \|\phi - \psi_j\|I_n) \\ &\preceq \Sigma_t^{\sigma_{j+}}(\psi_j) + g_t^{\sigma_{j+}}(\psi_j) \|\phi - \psi_j\| \\ &\preceq \psi_{t+j} + \alpha_1 \eta_1^t (\kappa_E + \beta_1) I_n, \end{aligned}$$

for all  $\phi \in E$ , where  $\kappa_E \triangleq \sup_{\phi \in E} \|\phi\|$ . This implies the existence of the desired constant  $\beta_E$ , which in turn guarantees the existence of the desired constants  $\alpha_E$  and  $\eta_E$  according to Theorem 1.  $\blacksquare$

The above lemma indicates that the covariance trajectories starting from any initial covariance in a bounded set  $E$  are bounded uniformly by  $\beta_E I_n$ . The bound  $\beta_E$  depends only on the underlying set  $E$  instead of the particular value of the initial covariance. Furthermore, the same bound also applies if we remove a finite number of steps from the schedule. We next use this result to show a key lemma of this subsection.

*Lemma 4 (Contraction):* Let  $j \in \mathbb{Z}_+$  be arbitrary.

(i) For any bounded set  $E \subseteq \mathcal{A}$ , there exists a finite integer  $l_0$  such that

$$\|\Sigma_l^{\sigma_{j+}}(\phi_1) - \Sigma_l^{\sigma_{j+}}(\phi_2)\| \leq c \|\phi_1 - \phi_2\|, \quad (16)$$

for all  $\phi_1, \phi_2 \in E$  and all  $l \geq l_0$ .

- (ii) There exists a finite integer  $l$  (possibly depending on  $\hat{\phi}$  and  $r$ ) such that  $\Sigma_l^{\sigma_{j+}}$  is a contraction on  $\mathcal{B}(\hat{\phi}; r)$  with contraction constant  $c$ , namely, it satisfies (16) for all  $\phi_1, \phi_2 \in \mathcal{B}(\hat{\phi}; r)$  and  $\mathcal{B}(\hat{\phi}; r)$  is invariant under  $\Sigma_l^{\sigma_{j+}}$ .
- (iii) For any  $0 < r_1 < r_2 < \infty$ , there exists a finite  $l \in \mathbb{Z}_+$  such that  $\Sigma_l^{\sigma_{j+}}$  is a contraction on both  $\mathcal{B}(\hat{\phi}; r_1)$  and  $\mathcal{B}(\hat{\phi}; r_2)$  with the same contraction constant  $c$ .

*Proof:* (i) Fix arbitrary  $\phi_1, \phi_2 \in E$ . By Theorem 1 and Lemma 3, we have

$$\begin{aligned}\Sigma_t^{\sigma^{j+}}(\phi_1) &\leq \Sigma_t^{\sigma^{j+}}(\phi_2 + \|\phi_1 - \phi_2\|I_n) \\ &\leq \Sigma_t^{\sigma^{j+}}(\phi_2) + \alpha_E \eta_E^t \|\phi_1 - \phi_2\|I_n,\end{aligned}$$

where  $\alpha_E$  and  $\eta_E$  are the constants mentioned in Lemma 3. Thus, there exists a finite integer  $l_0$  such that

$$\Sigma_l^{\sigma^{j+}}(\phi_1) \leq \Sigma_l^{\sigma^{j+}}(\phi_2) + c\|\phi_1 - \phi_2\|I_n, \forall l \geq l_0.$$

Similarly, we can show that

$$\Sigma_l^{\sigma^{j+}}(\phi_2) \leq \Sigma_l^{\sigma^{j+}}(\phi_1) + c\|\phi_2 - \phi_1\|I_n, \forall l \geq l_0.$$

(ii) Since  $\mathcal{B}(\hat{\phi}; r)$  is bounded, part (i) implies the existence of an  $l_0$  for which inequality (16) holds for all  $\phi_1, \phi_2 \in \mathcal{B}(\hat{\phi}; r)$ . Furthermore, since  $\hat{\phi} \in \mathcal{L}^\sigma$  is an accumulation point, there exists a finite integer  $l > l_0$ , such that

$$\|\Sigma_l^{\sigma^{j+}}(\phi) - \hat{\phi}\| \leq (1-c)r.$$

Therefore, for any  $\phi \in \mathcal{B}(\hat{\phi}; r)$ , we have

$$\begin{aligned}\|\Sigma_l^{\sigma^{j+}}(\phi) - \hat{\phi}\| &\leq \|\Sigma_l^{\sigma^{j+}}(\hat{\phi}) - \hat{\phi} + \Sigma_l^{\sigma^{j+}}(\phi) - \Sigma_l^{\sigma^{j+}}(\hat{\phi})\| \\ &\leq (1-c)r + c \cdot r = r,\end{aligned}$$

which implies that  $\mathcal{B}(\hat{\phi}; r)$  is invariant under the mapping  $\Sigma_l^{\sigma^{j+}}(\cdot)$ .

(iii) Let  $l_0$  be a constant such that (16) holds for all  $\phi_1, \phi_2 \in \mathcal{B}(\hat{\phi}; r_2)$ . Then, following the argument as in the proof of part (ii), we can show that the same  $l > l_0$  that makes  $\Sigma_l^\sigma$  a contraction on  $\mathcal{B}(\hat{\phi}; r_1)$  will guarantee that  $\Sigma_l^\sigma$  is a contraction on  $\mathcal{B}(\hat{\phi}; r_2)$  as well. ■

The following corollary highlights an important consequence of the above lemma.

*Corollary 3:* Let  $l$  be an integer satisfying the desired properties of part (ii) of Lemma 4. Then, for any  $\phi \in \mathcal{B}(\hat{\phi}; r)$ , we have  $\Sigma_{k,l}^\sigma(\phi) \in \mathcal{B}(\hat{\phi}; r)$ , for all  $k \in \mathbb{Z}_+$ .

*Proof:* The result holds trivially for  $k = 0$ . Suppose it is true for some general  $k \in \mathbb{Z}_+$ , i.e.,  $\Sigma_{k,l}^\sigma(\phi) \in \mathcal{B}(\hat{\phi}; r)$ , then

$$\Sigma_{(k+1)l}^\sigma(\phi) = \Sigma_l^{\sigma^{j_0+}}(\Sigma_{k,l}^\sigma(\phi)) \in \mathcal{B}(\hat{\phi}; r),$$

where  $j_0 := k \cdot l$  and the last step follows from the fact that  $\mathcal{B}(\hat{\phi}; r)$  is invariant under the mapping  $\Sigma_l^{\sigma^{j_0+}}$ . ■

Corollary 3 indicates an important property of a feasible schedule  $\sigma$ , namely, for any neighborhood  $\mathcal{B}(\hat{\phi}; r)$  around any accumulation point  $\hat{\phi} \in \mathcal{L}^\sigma$ , there always exists an  $l \in \mathbb{Z}_+$  such that the covariance trajectory under  $\sigma$  must return to the neighborhood  $\mathcal{B}(\hat{\phi}; r)$  every  $l$  steps. This is a key property that guarantees the existence of a suboptimal periodic schedule.

*Theorem 4 (Periodic Suboptimal Schedule):* For any  $\delta > 0$  and  $\phi \in \mathcal{A}$ , there exists a periodic schedule  $\bar{\sigma}$  with a finite period  $l \in \mathbb{Z}_+$ , such that

(i) (Exponential Convergence):  $\Sigma_{k,l}^{\bar{\sigma}}(\phi) \rightarrow P^*$  exponentially as  $k \rightarrow \infty$ , where  $P^*$  is a fixed point of the composite Riccati mapping  $\Sigma_l^{\bar{\sigma}}(\cdot)$ .

(ii) (Suboptimal Performance): The infinite-horizon cost of  $\bar{\sigma}$  is bounded from above by

$$\bar{J}_\infty(\bar{\sigma}; \phi) \leq \bar{V}^*(\phi) + \delta.$$

*Proof:* Let  $\sigma^*$  be an optimal infinite-horizon schedule and let  $\phi^*$  be an accumulation point in  $\mathcal{L}^{\sigma^*}$ . According to Lemma 4, for any  $0 < r_1 < r_2 < \infty$ , there exists an  $l \in \mathbb{Z}_+$  for which  $\Sigma_l^{\sigma^{j+}}$  is a contraction on  $\mathcal{B}(\phi^*; r_i)$ ,  $i = 1, 2$ , with contraction constant  $c$  for all  $j \in \mathbb{Z}_+$ . Divide the schedule  $\sigma^*$  into a sequence of  $l$ -horizon sub-schedules and denote by  $\sigma_l^{(k)}$  the  $(k+1)^{th}$  sub-schedule for  $k \in \mathbb{Z}_+$ , i.e.,

$$\sigma_l^{(k)} = \{\sigma^*(k \cdot l), \sigma^*(k \cdot l + 1), \dots, \sigma^*((k+1) \cdot l - 1)\}.$$

By Lemma 4 and the Banach fixed point theorem, we know that  $\sigma_l^{(k)}$  has a unique fixed point in  $\mathcal{B}(\phi^*; r_1)$  for all  $k \in \mathbb{Z}_+$ . Define

$$\begin{aligned}\mathbb{M}_c^l &\triangleq \{\sigma_l \in \mathbb{M}^l : \Sigma_l^{\sigma_l}(\cdot) \text{ is a contraction on } \mathcal{B}(\phi^*; r_i) \\ &\quad i = 1, 2, \text{ with contraction constant } c\}.\end{aligned}$$

Clearly, the set  $\mathbb{M}_c^l$  is non-empty as  $\sigma_l^{(k)} \in \mathbb{M}_c^l$  for all  $k \in \mathbb{Z}_+$ . By the Banach fixed point theorem, for any  $\sigma_l \in \mathbb{M}_c^l$ , the composite Riccati mapping  $\Sigma_l^{\sigma_l}(\cdot)$  has a fixed point in  $\mathcal{B}(\phi^*; r_1)$ . Denote this fixed point by  $\Gamma(\sigma_l)$ . Define

$$\sigma_l^* \triangleq \arg \min_{\sigma_l \in \mathbb{M}_c^l} J_l(\Gamma(\sigma_l); \sigma_l),$$

The goal now is to show that the  $l$ -periodic schedule defined by:

$$\bar{\sigma} \triangleq \{\sigma_l^*, \sigma_l^*, \dots\},$$

is a suboptimal schedule with the desired properties.

To show property (i), we choose  $r_2$  large enough so that  $\phi \in \mathcal{B}(\phi^*; r_2)$ . Then the result follows directly from the contraction mapping theorem.

To prove the second property, we let  $P_k \triangleq \Gamma(\sigma_l^{(k)})$  and  $P^* \triangleq \Gamma(\sigma_l^*)$ . Since  $\sigma_l^{(k)} \in \mathbb{M}_c^l$  for all  $k \in \mathbb{Z}_+$ , we have

$$J_l(P^*; \sigma_l^*) \leq J_l(P_k; \sigma_l^{(k)}), \forall k \in \mathbb{Z}_+,$$

Let  $\psi_t^* = \Sigma_t^{\sigma^*}(P^*)$  be the optimal covariance trajectory under  $\sigma^*$  with initial covariance  $P^* \in \mathcal{B}(\phi^*; r_1)$ . By Corollary 3, we know that  $\psi_{k,l}^* \in \mathcal{B}(\phi^*; r_1)$  for all  $k \in \mathbb{Z}_+$ . Hence,

$$\|\psi_{k,l}^* - P_k\| \leq 2r_1, \quad \forall k \in \mathbb{Z}_+.$$

Therefore, for any  $k \in \mathbb{Z}_+$ , we have

$$\begin{aligned}J_l(P^*; \sigma_l^*) &\leq \sum_{t=1}^l \text{tr} \left( \Sigma_t^{\sigma_l^{(k)}}(P_k) \right) \\ &\leq \sum_{t=1}^l \left[ \text{tr} \left( \Sigma_t^{\sigma_l^{(k)}}(\psi_{k,l}^*) \right) + 2r_1 \cdot \text{tr} \left( g_t^{\sigma_l^{(k)}}(\psi_{k,l}^*) \right) \right] \\ &\leq \sum_{t=k \cdot l + 1}^{(k+1) \cdot l} [\text{tr}(\psi_t^*) + 2r_1 \alpha_{r_2}] \end{aligned}$$

where  $\alpha_{r_2}$  denotes the constant  $\alpha_E$  introduced in Lemma 3 when  $E = \mathcal{B}(\phi^*; r_2)$ . After some simple computations, the above inequality leads to

$$\bar{J}_\infty(\tilde{\sigma}; P^*) = \bar{J}_l(\sigma_l^*; P^*) \leq \bar{J}_\infty(\sigma^*; P^*) + 2r_1\alpha_{r_2}.$$

Since  $r_1$  is arbitrary and can be chosen independently of  $r_2$ , the term  $2r_1\alpha_{r_2}$  can be made arbitrarily small, which proves the result when initial covariance is  $P^*$ . The result also holds when  $P^*$  is changed to an arbitrary  $\phi \in \mathcal{A}$  due to Lemma 1. ■

Theorem 4 reveals several fundamental properties of the optimal solution of Problem 1 and the corresponding optimal trajectory of the covariance matrix. It shows that the optimal infinite-horizon cost can be approximated arbitrarily closely by a periodic schedule with a finite period. It also indicates that under this periodic schedule, the trajectory of the covariance matrix converges exponentially to a limit circle. Let  $\tilde{\sigma}$  be a periodic suboptimal schedule with period  $l \in \mathbb{Z}_+$ , which satisfies all the properties in Theorem 4. Let  $P^*$  be the fixed point of the mapping  $\Sigma_l^{\tilde{\sigma}}(\cdot)$ . Define

$$\mathcal{C} := \{P^*, \Sigma_1^{\tilde{\sigma}}(P^*), \Sigma_2^{\tilde{\sigma}}(P^*), \dots, \Sigma_{l-1}^{\tilde{\sigma}}(P^*)\}. \quad (17)$$

Then by Theorem 4, the trajectory  $\Sigma_t^{\tilde{\sigma}}(\phi)$  converges exponentially to the limit circle  $\mathcal{C}$ . Furthermore, the  $N$ -horizon average-per-stage cost  $\bar{J}_N(\tilde{\sigma}; \phi)$  converges to the average cost within the limit circle, namely,  $\frac{1}{l} \sum_{P \in \mathcal{C}} \text{tr}(P)$ .

## V. DISCUSSIONS AND CONCLUSIONS

The theoretical results derived in the last section provide us valuable insights about the infinite-horizon sensor scheduling problem. Theorem 4 motivates us to focus on the periodic schedules in solving the problem. The discussion in Section IV-A indicates that one can always evaluate the performance of a periodic schedule by directly starting from its fixed point. For example, the cost of the periodic schedule  $\{1, 2, 2, 1, 2, 2, \dots\}$  is

$$\frac{1}{3} [\text{tr}(P^* + \rho_1(P^*) + \rho_2(\rho_1(P^*)))],$$

where  $P^*$  satisfies the equation  $\rho_2(\rho_2(\rho_1(P^*))) = P^*$ , which can be efficiently computed using the contraction mapping algorithm. Therefore, one can easily evaluate the infinite-horizon performance of a periodic schedule. A straightforward way to solve Problem 1 is to first find the best  $l$ -periodic schedule by enumerating all the possible  $l$ -horizon sequences, and then gradually increase the period length until the performance no longer improves. Theorem 4 guarantees that one can approach the optimal cost arbitrarily closely using this approach. Although the complexity of this approach grows exponentially as  $l$  increases, it is still a reasonable solution procedure because a schedule with a large period is difficult to implement and is thus not preferred in practice.

In general, we envision the theoretical results derived in this paper being useful for the design and analysis of various infinite-horizon sensor scheduling algorithms. An important direction for further research is to establish conditions under

which the optimal finite-horizon average-per-stage cost  $\bar{V}_N$  will converge to the optimal infinite-horizon average-per-stage cost  $\bar{V}^*$  as  $N \rightarrow \infty$ .

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