On Piecewise Quadratic Control-Lyapunov Functions for Switched Linear Systems

Wei Zhang, Alessandro Abate, Michael P. Vitus and Jianghai Hu

***Abstract*— In this paper, we prove that a discrete-time switched linear system is exponentially stabilizable *if and only if* there exists a stationary hybrid-control law that consists of a homogeneous switching-control law and a piecewise-linear continuous-control law under which the closed-loop system has a *piecewise quadratic* Lyapunov function. Such a converse control-Lyapunov function theorem justifies many of the earlier controller-synthesis methods that have adopted piecewise-quadratic Lyapunov functions and piecewise-linear continuous-control laws for convenience or heuristic reasons. Furthermore, several important properties of the proposed stabilizing control law are derived and their connections to other existing controllers studied in the literature are discussed.**

I. Introduction

The stabilization problem of switched systems, especially autonomous switched linear systems, is receiving increasing research attention in recent years ([1], [2]). Many existing results approach the problem by searching for a switching strategy and a Lyapunov or Lyapunov-like function with decreasing values along the closed-loop system trajectory ([3], [4], [5], [6]). The main idea is first to parameterize the switching strategy and the Lyapunov-like function in terms of certain matrices and then to translate the Lyapunov or multiple-Lyapunov function theorem into matrix inequalities. If the solution of the matrix inequalities defines a quadratic common Lyapunov function under the proposed switching strategy, then the system is called *quadratic stabilizable*. It is proved in [3], [7] that the quadratic stabilizability is equivalent to the strict completeness of a certain set of symmetric matrices. From a different perspective, in [8], [9], it is shown that the system is quadratic stabilizable if there exists a stable convex combination of the subsystem matrices. The main limitation of these results is their conservatism. Many switched linear systems are asymptotically or exponentially stabilizable without having a quadratic common Lyapunov function ([2]). In [4], a piecewise quadratic structure is adopted for the Lyapunov function. By taking a so-called "largest-region-function switching strategy", the stabilization problem is formulated as a bilinear matrix inequality (BMI) problem and some heuristics are proposed to solve the BMI problem numerically.

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W. Zhang and J. Hu are with the School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN47906. Email:{zhang70,jianghai@purdue.edu}.

A. Abate and M. P. Vitus are with the Department of Aeronautics and Astronautics, Stanford University, CA 94305. Email:{aabate,vitus@stanford.edu}.

Recently, stabilization of nonautonomous switched linear systems through both switching control and continuous control has also been studied ([6], [10], [11], [12]). The methods are mostly direct extensions of the switching stabilization results for autonomous systems. By associating to each subsystem a feedback gain and a quadratic Lyapunov function, the stabilization problem is also formulated as a matrix inequality problem, where the feedback-gain matrices are part of the design variables.

The extensive use of various Lyapunov functions has sparked a great interest in the study of the converse Lyapunov function theorems for switched linear systems. In [13], [14], it is proved that the exponential stability of a switched linear system under *arbitrary switching* is equivalent to the existence of a piecewise quadratic, or a piecewise linear, or a smooth homogeneous common Lyapunov function. A converse control-Lyapunov function theorem is also derived in [15] for a switching-stabilizable uncertain switched linear system. Although the piecewise quadratic Lyapunov function has been widely used in studying the stabilization problem, its existence has not been proved for general exponentially stabilizable switched linear systems.

Despite the extensive literature in this field, some fundamental questions regarding the stabilization of a switched linear system remain open. As stated in [16], "necessary and sufficient conditions for the existence of a general (not necessarily quadratic) stabilizing feedback strategy are not known". In this paper, we derive an answer to this open problem. Our main contribution is the proof of the **equivalence** of the following statements for a discrete-time switched linear system:

(i) *The system is exponentially stabilizable;*
(ii) *There exists a piecewise-quadratic control-Lyapunov function;*
(iii) *There exists a stationary exponentially-stabilizing hybrid-control law that consists of a homogeneous switching-control law and a piecewise-linear continuous-control law.*

The equivalence of the above statements constitutes a converse *piecewise-quadratic control-Lyapunov function theorem* (Theorem 2), which has not been shown yet in the literature. Furthermore, this result also guarantees that to study the stabilization problem, it suffices to only consider the control-Lyapunov functions of piecewise-quadratic form and the continuous-control laws of piecewise-linear form. This justifies many of the earlier controller-synthesis methods that have adopted these forms for convenience or heuristic reasons.

This paper is organized as follows. The main results are stated as Theorem 2 and Theorem 3 in Section II. Then, the two theorems are proved in Sections III and IV, respectively. Some concluding remarks are given in Section V.

II. Problem Statement

We consider the discrete-time switched linear systems described by:

$$x(t+1) = A_{v(t)}x(t) + B_{v(t)}u(t),\ t \in \mathbb{Z}^+, \qquad (1)$$

where $\mathbb{Z}^+$ denotes the set of nonnegative integers, $x(t) \in \mathbb{R}^n$ is the continuous state, $v(t) \in \mathbb{M} \triangleq \{1, \ldots, M\}$ is the switching control that determines the discrete mode, and $u(t) \in \mathbb{R}^p$ is the continuous control. The sequence of pairs $\{(u(t), v(t))\}_{t=0}^{\infty}$ is called the *hybrid-control sequence*. For each $i \in \mathbb{M}$, A_i and B_i are constant matrices of appropriate dimensions and the pair (A_i, B_i) is called a subsystem.

The most general way of making a control decision is through the time-dependent (state-feedback) hybrid-control law, namely, the function $\xi_t \triangleq (\mu_t, \nu_t) : \mathbb{R}^n \to \mathbb{R}^p \times \mathbb{M}$ that maps each continuous state to a hybrid-control action that may vary with time t. Here, $\mu_t : \mathbb{R}^n \to \mathbb{R}^p$ and $\nu_t : \mathbb{R}^n \to \mathbb{M}$ are called the *(state-feedback) continuous-control law* and the *(state-feedback) switching-control law*, respectively, at time $t \in \mathbb{Z}^+$. A sequence of hybrid-control laws constitutes an *infinite-horizon feedback policy*: $\pi \triangleq \{\xi_0, \xi_1, \ldots, \ldots\}$. A policy $\pi = \{\xi, \xi, \ldots\}$ with the same control law $\xi_t = \xi$ at each time t is called a *stationary policy*. If system (1) is driven by a feedback policy π, then the closed-loop dynamics is governed by

$$x(t+1) = A_{\nu_t(x(t))}x(t) + B_{\nu_t(x(t))}\mu_t(x(t)),\ t \in \mathbb{Z}^+. \qquad (2)$$

The exponential stabilization problem is to find a policy π under which the trajectory $x(t)$ of system (2) originating from any initial state $x(0) = z$ satisfies[1]:

$$\|x(t)\|^2 \leq ac^t\|z\|^2,\ \forall t \in \mathbb{Z}^+, \qquad (3)$$

for some constants $a \geq 1$ and $0 < c < 1$, where $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^n$. If such a policy exists, then system (1) is called *exponentially stabilizable*. As a standard result of Lyapunov theory, a sufficient condition for the exponential stabilizability is the existence of the following Lyapunov function.

Theorem 1 ([17]): Suppose that there exist a policy π and a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}^+$ satisfying:

(i) $\kappa_1\|z\|^2 \leq V(z) \leq \kappa_2\|z\|^2$ for any $z \in \mathbb{R}^n$ and some finite positive constants κ_1 and κ_2;

(ii) $V(x(t)) - V(x(t+1)) \geq \kappa_3\|x(t)\|^2$ for any $t \in \mathbb{Z}^+$ and some constant $\kappa_3 > 0$, where $x(\cdot)$ is the closed-loop trajectory of system (2) under policy π.

Then system (1) is exponentially stabilizable by the policy π.

Definition 1: A nonnegative function $V : \mathbb{R}^n \to \mathbb{R}^+$ is called a *control-Lyapunov function* of system (1) if there exists a policy π such that V and π satisfy all the conditions in Theorem 1.

By Theorem 1, the existence of a control-Lyapunov function is a sufficient condition for the stabilizability of system (1). The main goal of this paper is to show that this condition is also necessary and more importantly, that the control-Lyapunov function can always be chosen to be piecewise quadratic and that the corresponding stabilizing policy can always be made stationary with a homogeneous switching-control law and a piecewise-linear continuous-control law. In other words, we shall prove the following theorems.

Theorem 2: System (1) is exponentially stabilizable if and only if there exists a piecewise-quadratic control-Lyapunov function, hereby referred to as PQCLF.

Theorem 3: System (1) is exponentially stabilizable (by an arbitrary feedback policy) if and only if it is exponentially stabilizable by a stationary feedback policy that consists of a homogeneous switching-control law and a piecewise-linear continuous-control law.

The proofs of Theorems 2 and 3 can be found in Sections III-C and IV-A, respectively.

III. A Converse PQCLF Theorem

This section is devoted to proving Theorem 2. The proof is based on a connection between the exponential stabilization problem and the switched LQR problem [18], [19]. Before proving the theorem, we first briefly review some of the key results for the switched LQR problem. Interested readers are referred to [18], [19] for an in-depth discussion on the switched LQR problem.

A. The Switched LQR Problem

Let $Q_i = Q_i^T \succ 0$ and $R_i = R_i^T \succ 0$ be the weighting matrices for the state and the control, respectively, for subsystem $i \in \mathbb{M}$. Define the running cost as

$$L(x, u, v) = x^T Q_v x + u^T R_v u, \qquad (4)$$

for $x \in \mathbb{R}^n$,$u \in \mathbb{R}^p$,$v \in \mathbb{M}$. Denote by $J_\pi(z)$ the total cost, possibly infinite, starting from $x(0) = z$ under policy π, i.e.,

$$J_\pi(z) = \sum\nolimits_{t=0}^{\infty} L(x(t), \mu_t(x(t)), \nu_t(x(t))). \qquad (5)$$

Define $V^*(z) = \inf_{\pi \in \Pi} J_\pi(z)$. Since the running cost is always nonnegative, the infimum always exists. The function $V^*(z)$ is called the *infinite-horizon value function*. It will be infinite if $J_\pi(z)$ is infinite for all the policies $\pi \in \Pi$. As a natural extension of the classical LQR problem, the *Discrete-time Switched LQR problem* (DSLQR) is defined as follows.

Problem 1 (DSLQR problem): For a given initial state $z \in \mathbb{R}^n$, find the infinite-horizon policy $\pi \in \Pi$ that minimizes $J_\pi(z)$ subject to equation (2).

Dynamic programming solves the DSLQR problem by introducing a sequence of value functions. Define the N-

[1]In this paper, the variable $z \in \mathbb{R}^n$ denotes a generic initial value of system (1).

horizon value function $V_N : \mathbb{R}^n \to \mathbb{R}$ as:

$$V_N(z) = \inf_{\substack{u(t)\in\mathbb{R}^p, v(t)\in\mathbb{M}\\ 0\le t\le N-1}} \left\{ \sum_{t=0}^{N-1} L(x(t),u(t),v(t)) \right| \text{subject to (1) with } x(0)=z \Big\}. \quad (6)$$

For any function $V : \mathbb{R}^n \to \mathbb{R}^+$ and any control law $\xi = (\mu,\nu) : \mathbb{R}^n \to \mathbb{R}^p \times \mathbb{M}$, denote by $\mathcal{T}_\xi$ the operator that maps V to another function $\mathcal{T}_\xi[V]$ defined as:

$$\mathcal{T}_\xi[V](z) = L(z,\mu(z),\nu(z)) + V(A_{\nu(z)}z + B_{\nu(z)}\mu(z)),\ \forall z \in \mathbb{R}^n. \quad (7)$$

Similarly, for any function $V : \mathbb{R}^n \to \mathbb{R}^+$, define the operator $\mathcal{T}$ by

$$\mathcal{T}[V](z) = \inf_{u\in\mathbb{R}^p, v\in\mathbb{M}} \big\{ L(z,u,v) + V(A_v z + B_v u) \big\},\ \forall z \in \mathbb{R}^n. \quad (8)$$

The equation defined above is called the *one-stage value iteration* of the DSLQR problem. We denote by $\mathcal{T}^k$ the composition of the mapping $\mathcal{T}$ with itself k times, i.e., $\mathcal{T}^k[V](z) = \mathcal{T}\big[\mathcal{T}^{k-1}[V]\big](z)$ for all $k \in \mathbb{Z}^+$ and $z \in \mathbb{R}^n$. Some standard results of Dynamic Programming are summarized in the following lemma.

Lemma 1 ([20]): Let $V_0(z) = 0$ for all $z \in \mathbb{R}^n$. Then

(i) $V_N(z) = \mathcal{T}^N[V_0](z)$ for all $N \in \mathbb{Z}^+$ and $z \in \mathbb{R}^n$;

(ii) $V_N(z) \to V^*(z)$ pointwise in $\mathbb{R}^n$ as $N \to \infty$.

(iii) The infinite-horizon value function satisfies the Bellman equation, i.e., $\mathcal{T}[V^*](z) = V^*(z)$ for all $z \in \mathbb{R}^n$.

(iv) If $R_v \succ 0$ for all $v \in \mathbb{M}$, then there exists a stationary optimal policy, i.e., there exists a hybrid-control law ξ^* such that $\mathcal{T}_{\xi^*}[V^*](z) = V^*(z)$, $\forall z \in \mathbb{R}^n$.

To derive the value function of the DSLQR problem, we introduce a few definitions. Denote by $\rho_i : \mathcal{A} \to \mathcal{A}$ the *Riccati Mapping* of subsystem $i \in \mathbb{M}$, i.e.,

$$\rho_i(P) = Q_i + A_i^T P A_i - A_i^T P B_i (R_i + B_i^T P B_i)^{-1} B_i^T P A_i. \quad (9)$$

Definition 2: Let $2^{\mathcal{A}}$ be the power set of $\mathcal{A}$. The mapping $\rho_{\mathbb{M}} : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$ defined by: $\rho_{\mathbb{M}}(\mathcal{H}) = \{\rho_i(P) : i \in \mathbb{M} \text{ and } P \in \mathcal{H}\}$ is called the *Switched Riccati Mapping* associated with Problem 1.

Definition 3: The sequence of sets $\{\mathcal{H}_k\}_{k=0}^N$ generated iteratively by $\mathcal{H}_{k+1} = \rho_{\mathbb{M}}(\mathcal{H}_k)$ with initial condition $\mathcal{H}_0 = \{0\}$ is called the *Switched Riccati Sets* associated with Problem 1.

The switched Riccati sets always start from a singleton set $\{0\}$ and evolve according to the switched Riccati mapping. For any finite N, the set $\mathcal{H}_N$ consists of up to M^N p.s.d. matrices. An important fact about the DSLQR problem is that its value functions are completely characterized by the switched Riccati sets.

Theorem 4 ([21]): The N-horizon value function for the DSLQR problem is given by

$$V_N(z) = \min_{P\in\mathcal{H}_N} z^T P z. \quad (10)$$

Remark 1: Clearly, for any finite N, the value function V_N is a piecewise quadratic function. It will be shown that if the system is exponentially stabilizable, then there must exist a finite N such that V_N is a control-Lyapunov function of system (1).

B. V^* as a Control-Lyapunov Function

It is a well-known result that if a linear time-invariant system is stabilizable, then the infinite-horizon value function of the corresponding classical LQR problem is a control-Lyapunov function. This subsection generalizes this result to the switched linear system case. We shall show that if system (1) is exponentially stabilizable, then the infinite-horizon value function V^* of the DSLQR problem must be a control-Lyapunov function of system (1).

We first introduce some notations. Denote by $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ the smallest and the largest eigenvalue of a p.s.d. matrix. Define

$$\lambda_Q^- = \min_{i\in\mathbb{M}}\{\lambda_{\min}(Q_i)\},\ \lambda_Q^+ = \max_{i\in\mathbb{M}}\{\lambda_{\max}(Q_i)\},$$
$$\lambda_R^- = \min_{i\in\mathbb{M}}\{\lambda_{\min}(R_i)\},\ \lambda_R^+ = \max_{i\in\mathbb{M}}\{\lambda_{\max}(R_i)\},$$
$$\sigma_A^+ = \max_{i\in\mathbb{M}}\left\{\sqrt{\lambda_{\max}(A_i^T A_i)}\right\}.$$

Denote by $I_B^+ \subset \mathbb{M}$ the set of indices of nonzero B matrices, i.e., $I_B^+ \triangleq \{i \in \mathbb{M} : \|B_i\| \neq 0\}$. Let $\sigma_{\min}^+(\cdot)$ be the smallest *positive* singular value of a nonzero matrix. If $I_B^+ \neq \emptyset$, define $\hat{\sigma}_B = \min_{i\in I_B^+}\{\sigma_{\min}^+(B_i)\}$. Since $R_v \succ 0$ for each $v \in \mathbb{M}$, by Lemma 1, there must exist a hybrid-control law ξ^* such that $\mathcal{T}_{\xi^*}[V^*](z) = V^*(z)$, $\forall z \in \mathbb{R}^n$. Then, the policy $\pi^* = \{\xi^*, \xi^*, \ldots\}$ is the *stationary optimal policy*.

Our first task is to relate the exponential stabilizability to the boundedness of the value function V^*. In particular, we want to show that the exponentially stabilizability implies that $V^*(z) \le \beta\|z\|^2$ for all $z \in \mathbb{R}^n$ and some constant $\beta < \infty$. The main challenge here is that the stabilizing policy may employ a continuous control sequence $u(t)$ whose norm does not converge to zero exponentially fast. Our strategy is to project out the component of each $u(t)$ that lies in the null space of $B_{v(t)}$ and show that the norm of its orthogonal part converges to zero exponentially fast. To this end, the following lemma is needed.

Lemma 2: Let $B \in \mathbb{R}^{n\times p}$ be arbitrary but $B \neq 0$. Then for any $u \in \mathbb{R}^p$ in the column space of B^T, i.e., $u \in col(B^T)$, we must have $\|u\| \le \|Bu\|/\sigma_{\min}^+(B)$.

Proof: The result follows immediately when B has a full column rank. Suppose that B is not full column rank. By the theory of singular value decomposition, there exists unitary matrices $U = [U_1, U_2]$ and $V = [V_1, V_2]$ such that

$$B = [U_1, U_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

Since the column space $col(B^T)$ is the orthogonal complement of the null space of B, we have $V_2^T u = 0$. Thus, $\|u\| = \|V^T u\| = \|V_1^T u\|$. Therefore,

$$\|Bu\|^2 = u^T V_1 \Sigma^2 V_1^T u \ge \sigma_{\min}^+(B)^2 \|V_1^T u\|^2 = \sigma_{\min}^+(B)^2 \|u\|^2.$$

Thus $\|u\| \leq \|Bu\|/\sigma^+_{\min}(B)$. ■

With the above lemma, we are able to relate the exponential stabilizability to the boundedness of V^*.

Lemma 3: Suppose that system (1) is exponentially stabilizable. Then there exists a positive constant $\beta < \infty$ such that $\lambda^-_Q \|z\|^2 \leq V^*(z) \leq \beta \|z\|^2$, for all $z \in \mathbb{R}^n$.

Proof: Let $z \in \mathbb{R}^n$ be arbitrary and fixed. Obviously, $V^*(z)$ can be no smaller than the one-step state cost, which implies $V^*(z) \geq \lambda^-_Q \|z\|^2$. To prove that $V^*(z) \leq \beta \|z\|^2$, let $\pi = \{(\mu_t, \nu_t)\}_{t=0}^\infty$ be an exponentially stabilizing policy. By (3), the closed-loop trajectory $x(t)$ with initial condition $x(0) = z$ satisfies $\|x(t)\|^2 \leq ac^t \|z\|^2$, for some $a \in [1, \infty)$ and $c \in (0, 1)$. Thus, $\sum_{t=0}^\infty \|x(t)\|^2 \leq \frac{a}{1-c} \|z\|^2$. Denote by $(u(t), v(t))$ the hybrid-control sequence generated by π, i.e., $u(t) = \mu_t(x(t))$ and $v(t) = \nu_t(x(t))$. If $I^+_B = \emptyset$, then $u(t)$ can be chosen to be zero for each $t \geq 0$. Thus,

$$V^*(z) = \sum_{t=0}^\infty x^T(t) Q_{v(t)} x(t) \leq \frac{a\lambda^+_Q}{1-c} \|z\|^2,$$

which is the desired result with $\beta = \frac{a\lambda^+_Q}{1-c}$. We now suppose that $I^+_B \neq \emptyset$, which implies that $\hat{\sigma}_B > 0$. Define a new control sequence

$$\tilde{u}(t) = \begin{cases} 0, & \text{if } B_{v(t)} = 0, \\ [u(t)]_{B^T_{v(t)}}, & \text{otherwise,} \end{cases}$$

where $[\cdot]_{B^T_{v(t)}}$ denotes the projection of a given vector onto the column space of $B^T_{v(t)}$. Then $u(t) - \tilde{u}(t)$ is in the null space of $B_{v(t)}$, implying that $B_{v(t)}\tilde{u}(t) = B_{v(t)}u(t)$. As a result, under the new hybrid control sequence $(\tilde{u}(t), v(t))$, the closed-loop trajectory is still $x(t)$. Since $(\tilde{u}(t), v(t))$ is just one choice of the hybrid control sequence, we have

$$\begin{aligned} V^*(z) &\leq \sum_{t=0}^\infty L(x(t), \tilde{u}(t), v(t)) \\ &\leq \lambda^+_Q \frac{a}{1-c} \|z\|^2 + \lambda^+_R \sum\nolimits_{t=0}^\infty \|\tilde{u}(t)\|^2. \end{aligned} \tag{11}$$

Furthermore, by Lemma 2, we have

$$\begin{aligned} &\sum\nolimits_{t=0}^\infty \|\tilde{u}(t)\|^2 \leq \frac{1}{\hat{\sigma}^2_B} \sum\nolimits_{t=0}^\infty \|B_{v(t)}\tilde{u}(t)\|^2 \\ &= \frac{1}{\hat{\sigma}^2_B} \sum\nolimits_{t=0}^\infty \|B_{v(t)}u(t)\|^2 \\ &\leq \frac{1}{\hat{\sigma}^2_B} \sum\nolimits_{t=0}^\infty \|x(t+1) - A_{v(t)}x(t)\|^2 \\ &\leq \frac{2}{\hat{\sigma}^2_B} \left[\frac{ac}{1-c} + (\sigma^+_A)^2 \frac{a}{1-c} \right] \|z\|^2 \\ &\leq \frac{2a[c + (\sigma^+_A)^2]}{\hat{\sigma}^2_B (1-c)} \|z\|^2. \end{aligned}$$

This inequality together with (11) yields the desired result. ■

We now prove the main theorem of this subsection.

Theorem 5: If system (1) is exponentially stabilizable, then the infinite-horizon value function $V^*(z)$ is a control-Lyapunov function of system (1) with a stabilizing policy $\pi^* = \{\xi^*, \xi^*, \ldots\}$.

Proof: Suppose that system (1) is exponentially stabilizable. By Lemma 3, V^* satisfies the first condition of Theorem 1. By the definition of ξ^*, $V^*(z) = \mathcal{T}_{\xi^*}[V^*](z)$. This implies that

$$\begin{aligned} &V^*(z) - V^*(A_{\nu^*(z)}z + B_{\nu^*(z)}\mu^*(z)) \\ =&z^T Q_{\nu^*(z)} z + [\mu^*(z)]^T R_{\nu^*(z)} [\mu^*(z)] \\ >&\lambda^-_Q \|z\|^2. \end{aligned}$$

Hence, V^* is a control-Lyapunov function of system (1) with a stationary stabilizing policy $\pi^* = \{\xi^*, \xi^*, \ldots\}$. ■

By this theorem, whenever system (1) is exponentially stabilizable, the optimal policy π^* is stabilizing and $V^*(z)$ is a control-Lyapunov function. However, the function V^* may not be piecewise quadratic. To prove Theorem 2, in the next section we shall find an approximation of V^* which is piecewise quadratic yet close enough to V^* so that it remains a valid Lyapunov function.

C. Proof of Theorem 2

Since V^* is a control-Lyapunov function, roughly speaking, any function that is uniformly close to V^* will also be a control-Lyapunov function. By part (ii) of Lemma 1, the finite-horizon value function V_N, which is piecewise quadratic, converges pointwise to V^* as $N \to \infty$. This motivates us to use V_N to approximate V^* for large N. To guarantee that V_N will eventually become a Lyapunov function, we shall first ensure that the convergence of V_N to V^* is uniform on a compact set, say the unit ball.

Theorem 6 ([22]): If $V^*(z) \leq \beta \|z\|^2$ for some $\beta < \infty$, then

$$|V_{N_1}(z) - V_N(z)| \leq \alpha_\beta \gamma^N_\beta \|z\|^2, \tag{12}$$

for any $N_1 \geq N \geq 1$, where

$$\gamma_\beta = \tfrac{1}{1+\lambda^-_Q/\beta} < 1 \quad \text{and} \quad \alpha_\beta = \max\{1, \tfrac{\sigma^+_A}{\gamma_\beta}\}. \tag{13}$$

By this theorem, for large N, V^* can be approximated by V_N uniformly well on any compact set. As a result, the optimal control law ξ^* can also be approximated by ξ_N, which is defined by:

$$\begin{aligned} \xi_N(z) &= (\mu_N(z), \nu_N(z)) \\ &\triangleq \operatorname*{arg\,inf}_{u\in\mathbb{R}^p, v\in\mathbb{M}} \{L(z, u, v) + V_N(A_v z + B_v u)\} \end{aligned} \tag{14}$$

Let $\pi_N \triangleq \{\xi_N, \xi_N, \ldots\}$ be the stationary policy generated by V_N. Due to the convergence of V_N to V^*, the policy π_N will eventually become a stabilizing policy.

Proof: [Proof of Theorem 2] By Lemma 1 and equation (14), we know that $V_{N+1}(z) = \mathcal{T}_{\xi_N}[V_N](z)$, for all $z \in \mathbb{R}^n$. This implies that

$$\begin{aligned} &V_{N+1}(z) - V_N(A_{\nu_N(z)}z + B_{\nu_N(z)}\mu_N(z)) \\ &= z^T Q_{\nu_N(z)} z + \mu_N(z)^T R_{\nu_N(z)} \mu_N(z) \\ &> \lambda^-_Q \|z\|^2. \end{aligned} \tag{15}$$

By Lemma 3, the exponential stabilizability implies the existence of a positive constant $\beta < \infty$ such that $V^*(z) \leq$

$\beta\|z\|^2$, $\forall z \in \mathbb{R}^n$. Let γ_β and α_β be defined in terms of β as in (13). By Theorem 6, $V_{N+1}(z) \le V_N(z) + \alpha_\beta \gamma_\beta^N \|z\|^2$. Substituting this inequality into (15) yields

$$\begin{aligned} &V_N(z) - V(A_{\nu_N(z)} z + B_{\nu_N(z)} \mu_N(z)) \\ &\ge (\lambda_Q^- - \alpha_\beta \gamma_\beta^N)\|z\|^2. \end{aligned}$$

Since $\gamma_\beta < 1$ and $\lambda_Q^- > 0$, there must be a finite integer N_0 such that $(\lambda_Q^- - \alpha_\beta \gamma_\beta^N) > 0$ for all $N \ge N_0$. Therefore, for all $N \ge N_0$, the stationary policy π_N is exponentially stabilizing and V_N is a PQCLF. ■

The above proof is constructive. It not only shows the existence of a PQCLF, but also indicates that the stabilizing policy and the PQCLF can be chosen to be π_N and V_N, respectively. We point out this important fact in the following corollary.

Corollary 1: If system (1) is exponentially stabilizable, then there exists a finite integer N_0 such that for all $N \ge N_0$, V_N is a PQCLF of system (1) with a stationary stabilizing feedback policy π_N.

IV. The Stationary Stabilizing Feedback Policy

By Corollary 1, if system (1) is exponentially stabilizable, then it must be stabilizable by $\pi_N = \{\xi_N, \xi_N, \ldots\}$ for all large N. In this section, we will prove Theorem 3 and derive some important properties of the policy π_N.

A. Proof of Theorem 3

Due to the special structure of the value function as given in (10), the control law ξ_N defined in (14) can be characterized analytically.

Theorem 7: The control law defined in (14) is given by:

$$\begin{aligned} \xi_N(z) &= (\mu_N(z), \nu_N(z)) \\ &= \left(-K_{i_N(z)}\left(P_N(z)\right) \cdot z, i_N(z)\right), \end{aligned} \tag{16}$$

where $K_i(P)$ denotes the Kalman gain of subsystem i for a given p.s.d. matrix P, i.e.,

$$K_i(P) \triangleq (R_i + B_i^T P B_i)^{-1} B_i^T P A_i. \tag{17}$$

and

$$(P_N(z), i_N(z)) = \underset{P \in \mathcal{H}_N, i \in \mathbb{M}}{\arg\min}\; z^T \rho_i(P) z. \tag{18}$$

Proof: To find ξ_N, we need to solve the following optimization problem:

$$\begin{aligned} f(z) &\triangleq \inf_{u \in \mathbb{R}^p, i \in \mathbb{M}} \Big[\min_{P \in \mathcal{H}_N} u^T R_i u + z^T Q_i z \\ &\qquad + (A_i z + B_i u)^T P (A_i z + B_i u) \Big] \\ &= \min_{i \in \mathbb{M}, P \in \mathcal{H}_N} \Big\{ z^T Q_i z + \inf_{u \in \mathbb{R}^p} \big[u^T R_i u \\ &\qquad + (A_i z + B_i u)^T P (A_i z + B_i u) \big] \Big\}. \end{aligned} \tag{19}$$

For each $i \in \mathbb{M}$ and $P \in \mathcal{H}_N$, the quantity inside the square bracket is quadratic in u. Thus, the optimal value of u can be easily computed as $u^* = -K_i(P)z$, where $K_i(P)$ is the Kalman gain defined in (17). Substituting u^* into (19) and simplifying the resulting expression yields $f(z) = z^T \rho_{i_N(z)}(P_N(z)) z$, where $P_N(z)$ and $i_N(z)$ are defined in (18). ■

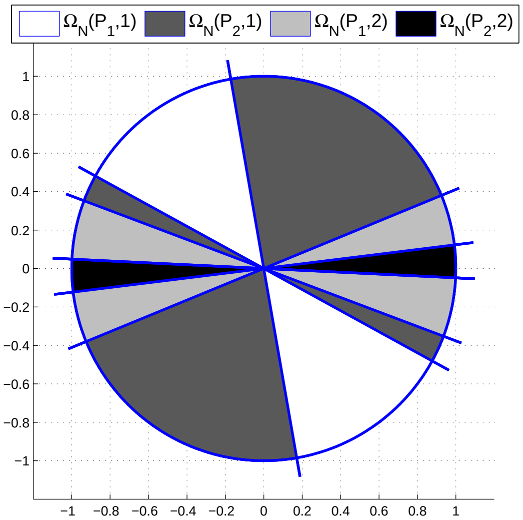

Fig. 1. Typical Decision regions

We now prove Theorem 3.

Proof: [Proof of Theorem 3] Since both $\mathcal{H}_N$ and $\mathbb{M}$ contain finitely many elements, the minimizer $(P_N(z), i_N(z))$ in (14) must be piecewise constant. Hence, by (16), we know that μ_N is piecewise linear and ν_N is homogeneous. This together with Corollary 1 implies Theorem 3. ■

B. Properties of ξ_N

For each pair $(P, i) \in \mathcal{H}_N \times \mathbb{M}$, define a subset of $\mathbb{R}^n$ as:

$$\Omega_N(P,i) = \{z \in \mathbb{R}^n : (P,i) = \underset{\hat{P} \in \mathcal{H}_N, \hat{i} \in \mathbb{M}}{\arg\min}\; z^T \rho_{\hat{i}}^0(\hat{P}) z\}. \tag{20}$$

The set $\Omega_N(P,i)$ such defined is called a *decision region* associated with ξ_N in the sense that the points within the same decision region correspond to the same pair of feedback gain $K_i(P)$ and switching control i under the control law ξ_N.

According to (20), a decision region must be a homogeneous cone. This implies that the control law ξ_N is also homogeneous. Furthermore, it follows immediately from (14) that the continuous-control law μ_N is piecewise linear with a constant feedback gain within each decision region. Note that a decision region $\Omega_N(P,i)$ may be disconnected except at the origin 0 and the union of all the decision regions covers the entire space $\mathbb{R}^n$. For example, if $\mathbb{M} = \{1, 2\}$ and $\mathcal{H}_N$ contains two matrices P_1 and P_2, then there will be four conic decision regions as shown in Fig. 1.

The decision regions that have the same switching control constitute a *switching region*. For each $i \in \mathbb{M}$, the switching region $S_N(i)$ is defined as:

$$S_N(i) = \cup_{P \in \mathcal{H}_N} \Omega_N(P,i). \tag{21}$$

The states that reside in the same switching region evolve through the same subsystem; however, they may be controlled by different feedback gains.

C. Relations to Other Controllers

Many hybrid-control laws proposed in the literature ([3], [4], [11]) can be written in the following form:

$$\begin{aligned} &\tilde{\xi}(z) = (\tilde{\mu}(z), \tilde{\nu}(z)) = (F_{\tilde{i}(z)} z, \tilde{i}(z)) \\ \text{with} \quad &\tilde{i}(z) = \arg\min_{i \in \mathbb{M}} z^T Q_i z, \end{aligned} \tag{22}$$

where $\{F_i\}_{i\in\mathbb{M}}$ are the feedback gains and $\{Q_i\}_{i\in\mathbb{M}}$ are some symmetric matrices characterizing the decision regions. The control law $\tilde{\xi}(z)$ is exponentially stabilizing if $\{F_i\}_{i\in\mathbb{M}}$ and $\{Q_i\}_{i\in\mathbb{M}}$ satisfy certain matrix inequalities. However, these matrix inequalities are only sufficient conditions for the exponential stabilizability. There may not be a stabilizing control law necessarily of the form (22) even when the switched linear system is exponentially stabilizable.

By a similar argument as in the last subsection, it can be easily verified that (i) $\tilde{\xi}$ divides the state space into at most M conic decision regions; (ii) each switching control is associated with only one feedback gain.

Compared with $\tilde{\xi}$, the control law ξ_N is more general. The number of decision regions of ξ_N may be larger than M and the same switching control may be paired with more than one feedback gains. *It is interesting to realize that these small differences are enough to render the structure of ξ_N necessary for the exponential stabilization of a switched linear system.*

V. Conclusions

This paper establishes a necessary and sufficient condition for the exponential stabilizability of switched linear systems. We have proved that a switched linear system is exponentially stabilizable if and only if the there exists a PQCLF and a stationary hybrid-control law that consists of a homogeneous switching-control law and a piecewise-linear continuous-control law. This existence result is rather useful for the design of stabilizing controllers. It allows us to only consider the control-Lyapunov functions of piecewise-quadratic form and the continuous-control laws of piecewise-linear form in studying the exponential stabilization problem of a switched linear system. Future research will focus on developing algorithms to efficiently compute a control-Lyapunov function and the corresponding stabilizing control law when the system is known to be exponentially stabilizable.

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